Benha University Faculty of Engineering- Shoubra Eng. Mathematics & Physics Department Preparatory Year	Final Term Exam Date: January 10 , 2017 Course: Mathematics 1 – A Duration: **3** hours

- The Exam consists of one page
- Answer **All** Questions
- No. of questions: 4
- Total Mark: 100

Question 1

(a)Find **y`** from the following: 12

(i) $y = x^{-3} + 4^{x^2} - 2$ (ii) $y = \sin x^2 . \sec x$ (iii) $y = \ln\ln x + \cos\ln x$

(iv) $\mathrm{y} = \sec^3 \mathrm{x} + \log \mathrm{x}$ (v) $y^3 + e^{xy} + x^x = 0$ (vi) $\mathrm{y} = \mathrm{t}\tan\mathrm{t},\ x = \mathrm{t}\ln\mathrm{t}$

(b)Find the following limits: 8

(i) $\lim\limits_{x\to\pi} \frac{\tan 2x}{\sin x}$ (ii) $\lim\limits_{x\to 0} \frac{\ln(1+2x)}{3^x - 4^x}$ (iii) $\lim\limits_{x\to\infty} \frac{x^2}{x^2 + \ln x}$ (iv) $\lim\limits_{x\to\infty} \frac{x^6 - 3^x}{x^5 + 4^x}$

Question 2

(a)Write the Maclurin's expansion of the function: $f(x) = x\,cos\,x^2$. 4

(b)State and discuss Rolle's theorem for $f(x) = \sqrt[3]{(x-1)^2}$ in interval [0, 2]. 5

(c)Sketch the curve of each function : $f(x) = x - \ln x$, $g(x) = \frac{1}{\sqrt{x^2-1}}$. 12

(d)Find the integrals: (i)$\int (\frac{1}{x^3} + \frac{1}{3^x})\, dx$ $(ii) \int (3^x - 4^x)^2\, dx$ $(iii) \int \frac{\cos t}{\sqrt{\sin t}} dt$ 9

Question 3

(a)Find the sum to n terms from the series : $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \cdots$ 5

Also, show that its sum to infinity is 0.25 .

(b)Prove that : $(1+2+3+\cdots+n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$, $n \in N$. 5

(c)Sum the series : $1 + x(1+x) + x^2(1+x+x^2) \ldots .\ to\ n\ terms.$ 5

(d)Find the coefficient of x^{15} in the expansion : $(1+x)^4(1-x)^{-4}$. 5

Question 4

(a)Resolve the fraction : $\frac{6x^2-8x+3}{(1-x)^3}$ into its partial fractions. 6

(b)Solve the equation : $z^2 = 1 + i\sqrt{3}$. 6

(c)Discuss the existence of the solution of the linear system :

$x - 2y + 3z = -2,\ -x + y - 2z = 3,\ 2x - y + 3z = 1$ 6

(d)Solve the equation $x^4 - 7x^3 + 21x^2 + kx + 30 = 0$ given that $1 + 2i$ is a root. Also, find the value of k . 6

(e)Find the eigenvalues and eigenvectors of the matrix : $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ 6

Good Luck *Dr. Mohamed Eid* *Dr. Fathi Abdsallam*

Model Answer

Answer of Question 1

(a)(i) $y` = -3x^{-4} + 4^{x^2}.\ln 4 . 2x - 0$ (ii) $y` = \sin x^2 . \sec x . \tan x + \cos x^2 . 2x. \sec x$

(iii) $y` = \frac{1}{\ln x}.\frac{1}{x} - \sin \ln x . \frac{1}{x}$ (iv) $y` = 3\sec^3 x . \tan x + \frac{1}{\ln 10}.\frac{1}{x}$

(v) $3y^2 . y` + e^{xy}(y + xy`) + e^{x\ln x}(1 + \ln x) = 0$ (vi) $y` = \frac{t.\sec^2 t + \tan t}{1 + \ln t}$

--*12-Marks*

(b)(i) $\lim_{x\to\pi} \frac{\tan 2x}{\sin x} = \frac{0}{0} = \lim_{x\to\pi} \frac{2\sec^2 2x}{\cos x} = -2$

(ii) $\lim_{x\to 0} \frac{\ln(1+2x)}{3^x - 4^x} = \frac{0}{0} = \frac{2}{\ln(3/4)}$

(iii) $\lim_{x\to\infty} \frac{x^2}{x^2 + \ln x} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{2x}{2x + \frac{1}{x}} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{4x}{4x} = 1$

(iv) $\lim_{x\to\infty} \frac{x^6 - 3^x}{x^5 + 4^x} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{\frac{x^6}{4^x} - \frac{3^x}{4^x}}{\frac{x^5}{4^x} + 1} = 0$

--*8-Marks*

Answer of Question 2

(a) $f(x) = x\cos x^2 = x\left[1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots\right] = x - \frac{x^5}{2!} + \frac{x^9}{4!} - \cdots$

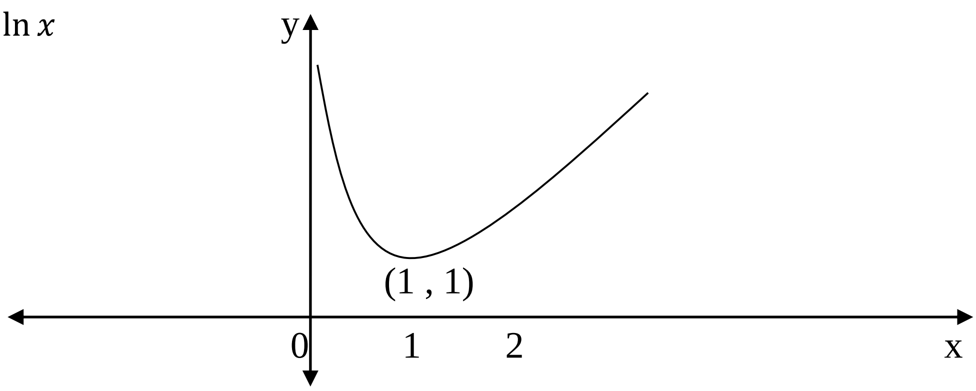
--*4-Marks*

(b)Rolle's theorem.

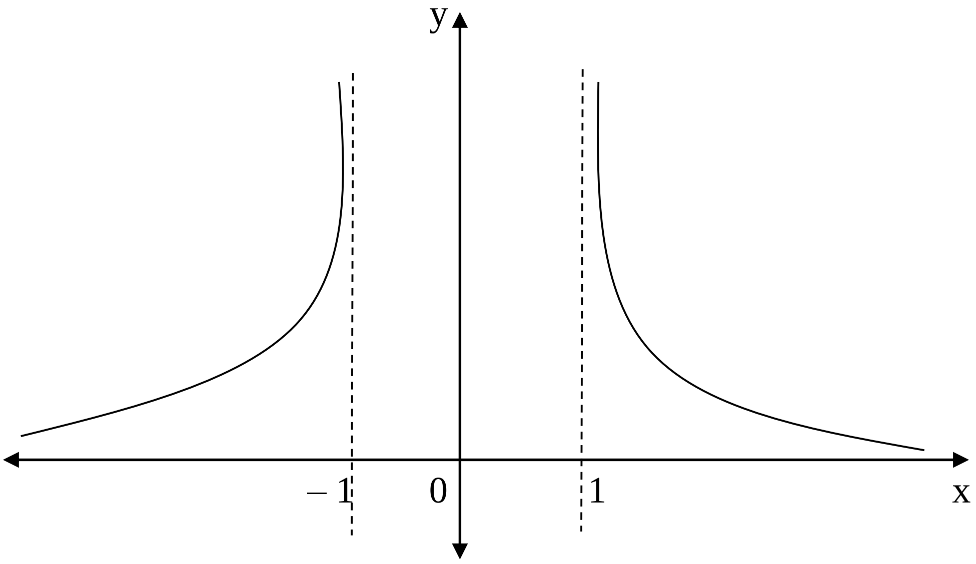
We see that $f(x)$ continuous in the given interval [0, 2] but its derivative $f`(x) = \frac{2}{3\sqrt[3]{x-1}}$ is infinity at 1 and f(0) = f(2) =1. The theorem is not satisfied.

--*4-Marks*

(c) $f(x) = x - \ln x$



$g(x) = \frac{1}{\sqrt{x^2-1}}$.



--*12-Marks*

$(d)(i) \int \left(\frac{1}{x^3} + \frac{1}{3^x}\right) dx = -\frac{1}{2}x^{-2} + \frac{\left(\frac{1}{3}\right)^x}{\ln(1/3)} + c$

$(ii) \int (3^x - 4^x)^2\, dx = \int (9^x + 16^x - 2\ \ 12^x) dx = \frac{9^x}{\ln 9} + \frac{16^x}{\ln 16} - 2\frac{12^x}{\ln 12} + c$

$(iii) \int \frac{\cos t}{\sqrt{\sin t}} dt = \int \cos t\, (\sin t)^{-1/2}\, dt = 2\sqrt{\sin t} + c$

--*9-Marks*

Dr. Mohamed Eid

<u>Answer of Question 3</u>

(a)Given the series $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots$ find its sum to n terms. Show that its sum to infinity is 0.25

Solution

$$u_r = \frac{1}{r(r+1)(r+2)}$$

$$\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} = \frac{(r+2)}{r(r+1)(r+2)} - \frac{r}{r(r+1)(r+2)} = \frac{2}{r(r+1)(r+2)} = 2u_r$$

$$u_r = \frac{1}{2}\left[\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right] = \frac{1}{2}[f(r) - f(r+1)]$$

$$S_n = \frac{1}{2}[f(1) - f(n+1)] = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right]$$

$$S_\infty = \lim_{n\to\infty} \frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right] = \frac{1}{4}$$

(b)Prove that $(1+2+3+...+n)^2 = 1^3+2^3+3^3+...+n^3$ for every integer $n \in \mathbb{N}$.

Solution

$$1^3+2^3+3^3+...+n^3 = \sum_{r=1}^{n} r^3 = \sum_{r=1}^{n} r(r+1)(r+2) - 3r(r+1) + r$$

$$= \frac{n(n+1)(n+2)(n+3)}{4} - \frac{3n(n+1)(n+2)}{3} + \frac{n(n+1)}{2}$$

$$= \frac{3n(n+1)(n+2)(n+3)}{12} - \frac{12n(n+1)(n+2)}{12} + \frac{6n(n+1)}{12}$$

$$= \frac{n(n+1)}{12}\left(3(n+2)(n+3) - 12(n+2) + 6\right)$$

$$= \frac{n(n+1)}{12}\left(3n^2 + 15n + 18 - 12n - 24 + 6\right)$$

$$= \frac{n(n+1)}{12}\left(3n^2 + 3n\right) = \frac{n^2(n+1)^2}{4} = \left[\frac{n(n+1)}{2}\right]^2 \quad = \left[\sum_{r=1}^{n} r\right]^2 = \left(1+2+3+...+n\right)^2$$

Another solution by (mathematical induction)

At n=1

$(1)^2 = 1^3$

At n=2

$(1+2)^2 = 1^3 + 2^3 = 9$

At n=k the statement is true

$(1+2+3+...+k)^2 = 1^3+2^3+3^3+...+k^3$

We prove that

$$(1+2+3+...+k+(k+1))^2 = 1^3+2^3+3^3+...+k^3+(k+1)^3$$

$$\text{L.H.S} = (1+2+3+...+k+(k+1))^2$$

$$= (1+2+3+...+k)^2 + (k+1)^2 + 2(k+1)(1+2+3+...+k)$$

$$= 1^3+2^3+3^3+...+k^3+(k+1)\left[(k+1) + 2\frac{k(k+1)}{2}\right]$$

$$= 1^3+2^3+3^3+...+k^3+(k+1)\left[(k+1)+k(k+1)\right]$$

$$= 1^3+2^3+3^3+...+k^3+(k+1)^3 = \text{R.H.S}$$

(c)Sum the series $1 + x(1+x) + x^2(1+x+x^2) + ...$ to n terms

Solution

$$1+x(1+x)+x^2(1+x+x^2)+\ldots\text{to n terms}$$

$$=1+x(1+x)+x^2(1+x+x^2)+\ldots+x^{n-1}(1+x+x^2\ldots+x^{n-1})$$

$$=1+x\,\frac{(1-x^2)}{1-x}+x^2\,\frac{(1-x^3)}{1-x}+\ldots+x^{n-1}\,\frac{(1-x^n)}{1-x}$$

$$=\frac{1}{1-x}\left[1-x+x(1-x^2)+x^2(1-x^3)+\ldots+x^{n-1}(1-x^n)\right]$$

$$=\frac{1}{1-x}\left[\left(1+x+x^2+\ldots+x^{n-1}\right)-\left(x+x^3+x^5+\ldots+x^{2n-1}\right)\right]$$

$$=\frac{1}{1-x}\left[\frac{1-x^n}{1-x}-\frac{x(1-x^{2n})}{1-x^2}\right]=\boxed{\frac{(1-x^n)(1-x^{n+1})}{(1-x)(1-x^2)}}$$

(d)Find coefficient x^{15} in the expansion $(1+x)^4(1-x)^{-4}$.

Solution

$$(1+x)^4(1-x)^{-4}=\left(1+4x+6x^2+4x^3+x^4\right)\left(\sum_{r=0}^{\infty}{}^{-4}C_r(-x)^r\right)$$

$$=\left(1+4x+6x^2+4x^3+x^4\right)\left(\sum_{r=0}^{\infty}(-1)^r\,C_r^{4+r-1}(-1)^r x^r\right)$$

$$=(1+4x+6x^2+4x^3+x^4)\left(\sum_{r=0}^{\infty}C_r^{3+r}x^r\right)$$

$$=(1+4x+6x^2+4x^3+x^4)\left(C_0^3+C_1^4x+C_2^5x^2+\ldots+C_n^{3+n}x^n+\ldots\right)$$

Coefficient x^{15} is

$$C_{15}^{18}+4C_{14}^{17}+6C_{13}^{16}+4C_{12}^{15}+C_{11}^{14}$$

$$=C_3^{18}+4C_3^{17}+6C_3^{16}+4C_3^{15}+C_3^{14}$$

$$=\frac{18.17.16}{3.2.1}+4.\frac{17.16.15}{3.2.1}6.\frac{16.15.14}{3.2.1}+4.\frac{15.14.13}{3.2.1}+\frac{14.13.12}{3.2.1}=9080$$

Answer of Question 4

(a)Resolve $\frac{6x^2-8x+3}{(1-x)^3}$ into its partial fraction.

Answer

$$\frac{6x^2-8x+3}{(1-x)^3}=\frac{A}{(1-x)^3}+\frac{B}{(1-x)^2}+\frac{C}{(1-x)}$$

$\therefore 6x^2 - 8x + 3 = A + B(1-x) + C(1-x)^2$ at $x = 1$ then $A = 1$

compare coefficent x^2 $6 = c$

compare coefficent x $-8 = -B - 2C$ *then* $B = -4$

$$\frac{6x^2 - 8x + 3}{(1-x)^3} = \frac{1}{(1-x)^3} + \frac{-4}{(1-x)^2} + \frac{6}{(1-x)}$$

(b) Solve the equation $z^2 = 1 + i\sqrt{3}$

Solution

The solution of the equation $z^2 = 1 + i\sqrt{3}$ is $z = \left(1 + i\sqrt{3}\right)^{1/2}$ first we put the number $1 + i\sqrt{3}$ in polar form $r = \sqrt{x^2 + y^2} = \sqrt{1+3} = 2$ and $\cos\theta = \frac{1}{2}, \sin\theta = \frac{\sqrt{3}}{2}$ which show that $\theta = \frac{\pi}{3}$ then $1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ hence

$$z = \left(1 + i\sqrt{3}\right)^{1/2} = \sqrt{2}\left[\cos(\tfrac{\pi}{3} + 2k\pi) + i\sin(\tfrac{\pi}{3} + 2k\pi)\right]^{1/2}$$

$$= \sqrt{2}\left[\cos(\tfrac{\pi + 6k\pi}{3}) + i\sin(\tfrac{\pi + 6k\pi}{3})\right]^{1/2}$$

$$= \sqrt{2}\left[\cos(\tfrac{\pi + 6k\pi}{6}) + i\sin(\tfrac{\pi + 6k\pi}{6})\right], \quad k = 0,1$$

At k=0

$$z_1 = \sqrt{2}\left[\cos(\tfrac{\pi}{6}) + i\sin(\tfrac{\pi}{6})\right] = \sqrt{2}\left[\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right] = \boxed{\frac{\sqrt{2}}{2}\left(\sqrt{3} + i\right)}$$

At k=1

$$z_2 = \sqrt{2}\left[\cos(\tfrac{\pi + 6\pi}{6}) + i\sin(\tfrac{\pi + 6\pi}{6})\right]$$

$$= \sqrt{2}\left[\cos(\tfrac{7\pi}{6}) + i\sin(\tfrac{7\pi}{6})\right] = \sqrt{2}\left[\frac{1}{2} - i\,\frac{\sqrt{3}}{2}\right] = \boxed{\frac{\sqrt{2}}{2}\left(-\sqrt{3} - i\right)}$$

(c)Discuss the existence of the solution for the system

$$x - 2y + 3z = -2$$
$$-x + y - 2z = 3$$
$$2x - y + 3z = 1$$

Solution

First write down the augmented matrix.

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{pmatrix}$$

We won't put down as many words in working this example. Here's the work for this augmented Matrix.

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{pmatrix} \xrightarrow[-2R_1+R_3\to R_3]{R_1+R_2\to R_2} \begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & -1 & 1 & 1 \\ 0 & 3 & -3 & 5 \end{pmatrix}$$

$$\xrightarrow{-2R_2} \begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -3 & 5 \end{pmatrix} \xrightarrow{-3R_2+R_3\to R_3} \begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

We won't go any farther in this example. Let's go back to equations to see why

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 8 \end{pmatrix} \Rightarrow \begin{array}{c} x-2y+3z=-2 \\ y-z=-1 \\ 0=8 \end{array}$$

$\operatorname{rank} A < \operatorname{rank} G$

Then the system inconsistent (no solution).

(d)Solve the equation $x^4-7x^3+21x^2+kx+30=0$ Given that $1+2i$ is a root of the equation, find the values k.

Solution

Since $w=1+2i$ is a root then $\overline{w}=1-2i$ is also root

Let the others roots are a and b

$(1+2i)+(1-2i)+a+b=7$

$2+a+b=7$

$a+b=5$ (1)

$ab(1+2i)(1-2i)=30$

$5ab=30$

$ab=6$ (4)

From (1) and (2) a and b are the roots of the equation $y^2-5y+6=0$ then

$a=2 \quad b=3$

To find k

$f(2) = (2)^4 - 7(2)^3 + 21(2)^2 + 2k + 30 = 0$

$2k + 74 = 0$

$\boxed{k = -37}$

(e)Find the eigen values and eigen vectors for the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

Solution:

Let the eigen vector in the form $X = \begin{bmatrix} x \\ y \end{bmatrix}$ we solve the equation $AX = \lambda X$ i.e.

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

which gives the system of equations

$x + 2y = \lambda x$ **or** $(\lambda - 1)x - 2y = 0$ (i)

$3x + 2y = \lambda y$ **or** $-3x + (\lambda - 2y) = 0$ (ii)

which has a solution subject to

$$\begin{bmatrix} (\lambda - 1) & -2 \\ -3 & (\lambda - 2) \end{bmatrix} = (\lambda - 4)(\lambda + 1) = 0 \Rightarrow \lambda = -1,\ \lambda = 4$$

when $\lambda = 4$ we use (i),(ii) then $3x - 2y = 0$ then

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is the first eigen vector corresponding the eigen

value $\lambda = 4$ and when $\lambda = -1$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Dr. Fathi Abdessalam